GEOMETRY OF ALMOST CLIFFORDIAN MANIFOLDS: CLASSES OF SUBORDINATED CONNECTIONS

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Abstract. An almost Clifford and an almost Cliffordian manifold is a G–structure based on the definition of Clifford algebras. An almost Clifford manifold based on $\mathcal{O} := \mathcal{C}l(s,t)$ is given by a reduction of the structure group $GL(km,\mathbb{R})$ to $GL(m,\mathcal{O})$, where $k = 2^{s+t}$ and $m \in \mathbb{N}$. An almost Cliffordian manifold is given by a reduction of the structure group to $GL(m,\mathcal{O})GL(1,\mathcal{O})$. We prove that an almost Clifford manifold based on $\mathcal{O}$ is such that there exists a unique subordinated connection, while the case of an almost Cliffordian manifold based on $\mathcal{O}$ is more rich. A class of distinguished connections in this case is described explicitly.

1. Introduction

First, let us recall some facts about a G-structures and their prolongations. There are two definitions of G–structures. The first reads that a G–structure is a principal bundle $P \to M$ with structure group G together with a soldering form θ. The second reads that it is a reduction of the frame bundle P^1M to the Lie group G. In the latter case, the soldering form θ is induced from a canonical soldering form on the frame bundle.

Now let $\mathfrak{g}$ be the Lie algebra of the Lie group G and let $\mathbb{V}$ be a vector space. From the structure theory we know that there is a G-invariant complement $\mathcal{D}$ of $\partial(\mathfrak{g} \otimes \mathbb{V}^*)$ in $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$, where ∂ is the operator of alternation, see [6]. Let us recall that the torsion of a linear connection lies in the space $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$.

The almost Clifford and almost Cliffordian structures are G-structures based on Clifford algebras. Two most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{C}l(0,2)$. An important geometric property of almost hypercomplex structure reads that there is no nontrivial G-invariant subspace $\mathcal{D}$ in $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$, because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = \mathbb{V}^*$, see [1]. For these reasons, in the latter case, there exists a distinguished class of linear connections compatible with the structure. Our goal is to describe some of these connections for almost Cliffordian G–structures based on Clifford algebras $\mathcal{C}l(s,t)$ generally.

2. Clifford algebras

The pair $(\mathbb{V}, Q)$, where $\mathbb{V}$ is a vector space of dimension n and Q is a quadratic form is called a quadratic vector space. To define Clifford algebras in coordinates,

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we start by choosing a basis $e_i,\ i=1,...,n$ of $\mathbb{V}$ and by $I_i,\ i=1,...,n$ we denote the image of e_i under the inclusion $\mathbb{V}\hookrightarrow \mathcal{C}l(\mathbb{V},Q)$. Then the elements I_i satisfy the relation

$$I_jI_k+I_kI_j=-2B_{jk}1,$$

where 1 is the unity in the Clifford algebra and B is a bilinear form obtained from Q by polarization. In a quadratic finite dimensional real vector space it is always possible to choose a basis e_i for which the matrix of the bilinear form B has the form

$$\begin{pmatrix} O_r & & \\ & E_s & \\ & & -E_t \end{pmatrix},\ r+s+t=n,$$

where E_k denotes the $k\times k$ identity matrix and O_k the $k\times k$ zero matrix. Let us restrict to the case $r=0$, whence B is nondegenerate. Then B defines inner product of signature (s,t) and we call the corresponding Clifford algebra $\mathcal{C}l(s,t)$. For example, $\mathcal{C}l(0,2)$ is generated by I_1,I_2, satisfying $I_1^2=I_2^2=-E$ with $I_1I_2=-I_2I_1$, i.e. $\mathcal{C}l(0,2)$ is isomorphic to $\mathbb{H}$.

Following the classification of the Clifford algebra, Boot periodicity reads that $\mathcal{C}l(0,n)\cong\mathcal{C}l(0,q)\otimes\mathbb{R}(16p)$, where $n=8p+q$, $q=0,\dots,7$ and $\mathbb{R}(N)$ denotes the $N\times N$ matrices with coefficients in $\mathbb{R}$. To determine explicit matrix representations we use the periodicity conditions

$$\begin{aligned}\mathcal{C}l(0,n)&\cong\mathcal{C}l(n-2,0)\otimes\mathcal{C}l(0,2),\\ \mathcal{C}l(n,0)&\cong\mathcal{C}l(0,n-2)\otimes\mathcal{C}l(2,0),\\ \mathcal{C}l(s,t)&\cong\mathcal{C}l(s-1,t-1)\otimes\mathcal{C}l(1,1)\end{aligned}$$

together with the explicit matrix representations of $\mathcal{C}l(0,2)$, $\mathcal{C}l(2,0)$, $\mathcal{C}l(1,0)$ and $\mathcal{C}l(0,1)$. More precisely, the identification of Clifford algebra $\mathcal{C}l(s,t)$, where either s or t is greater than 2, can be obtained by one of the following possibilities:

a) If $s>t$ then

$$\mathcal{C}l(s,t)\cong\mathcal{C}l(s-t,0)\otimes\overset{t}{\otimes}\mathcal{C}l(1,1).$$

Further, to classify the algebra $\mathcal{C}l(s-t,0)$, the following four cases are possible:

$$\begin{aligned}\mathcal{C}l(4p,0)&\cong\overset{p}{\otimes}(\mathcal{C}l(0,2)\otimes\mathcal{C}l(2,0)),\\ \mathcal{C}l(4p+1,0)&\cong\mathcal{C}l(1,0)\otimes\overset{p}{\otimes}(\mathcal{C}l(0,2)\otimes\mathcal{C}l(2,0)),\\ \mathcal{C}l(4p+2,0)&\cong\mathcal{C}l(2,0)\otimes\overset{p}{\otimes}(\mathcal{C}l(0,2)\otimes\mathcal{C}l(2,0)),\\ \mathcal{C}l(4p+3,0)&\cong\mathcal{C}l(0,1)\otimes\mathcal{C}l(2,0)\otimes\overset{p}{\otimes}(\mathcal{C}l(0,2)\otimes\mathcal{C}l(2,0)).\end{aligned}$$

b) If $s<t$ then

$$\mathcal{C}l(s,t)\cong\mathcal{C}l(0,t-s)\otimes\overset{s}{\otimes}\mathcal{C}l(1,1).$$

To classify the algebra $\mathcal{C}l(0,t-s)$, the following four cases are possible:

$$\begin{aligned}
\mathcal{C}l(0,4p) &\cong \overset{p}{\otimes}(\mathcal{C}l(2,0)\otimes\mathcal{C}l(0,2)),\\
\mathcal{C}l(0,4p+1) &\cong \mathcal{C}l(0,1)\otimes\overset{p}{\otimes}(\mathcal{C}l(2,0)\otimes\mathcal{C}l(0,2)),\\
\mathcal{C}l(0,4p+2) &\cong \mathcal{C}l(0,2)\otimes\overset{p}{\otimes}(\mathcal{C}l(2,0)\otimes\mathcal{C}l(0,2)),\\
\mathcal{C}l(0,4p+3) &\cong \mathcal{C}l(1,0)\otimes\mathcal{C}l(0,2)\otimes\overset{p}{\otimes}(\mathcal{C}l(2,0)\otimes\mathcal{C}l(0,2)).
\end{aligned}$$

c) If $s=t$ then

$$\mathcal{C}l(s,t)\cong\overset{s}{\otimes}\mathcal{C}l(1,1).$$

For example

$$\mathcal{C}l(3,0)\cong\mathcal{C}l(0,1)\otimes\mathcal{C}l(2,0),$$

where the matrix representation of $\mathcal{C}l(0,1)$ is given by the matrices

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ and }\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and the matrix representation of $\mathcal{C}l(2,0)$ is given by the matrices

$$E, I_1=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, I_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$I_3=I_1I_2=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

where E is an identity matrix. Now, the matrix representation of $\mathcal{C}l(3,0)$ is given by

$$\begin{pmatrix}E&0\\0&E\end{pmatrix},\begin{pmatrix}I_1&0\\0&I_1\end{pmatrix},\begin{pmatrix}I_2&0\\0&I_2\end{pmatrix},\begin{pmatrix}I_3&0\\0&I_3\end{pmatrix},$$
$$\begin{pmatrix}0&E\\-E&0\end{pmatrix},\begin{pmatrix}0&I_1\\-I_1&0\end{pmatrix},\begin{pmatrix}0&I_2\\-I_2&0\end{pmatrix},\begin{pmatrix}0&I_3\\-I_3&0\end{pmatrix},$$

for explicit description see [5].

We now focus on the algebra $\mathcal{O}:=\mathcal{C}l(s,t)$, i.e. the algebra generated by complex unities $I_i, i=1,\dots,t$ and product unites $J_j,\ j=1,\dots,s$, which are anti commuting, i.e. $I_i^2=-E$, $J_j^2=E$ and $K_iK_j=-K_jK_i$, $i\neq j$, where $K\in\{I_i,J_j\}$. On the other hand, this algebra is generated by elements F_i, $i=1,\dots,k$ as a vector space. We chose a basis F_i, $i=1,\dots,k$, such that $F_1=E$, $F_i=I_{i-1}$ for $i=2,\dots,t+1$, $F_j=J_{j-t-1}$ for $j=t+2,\dots,s+t+1$ and by all different multiples of I_i and J_j of length 2, ..., $s+t$. Let us note that both complex and product unities can be found among these multiple generators.

Lemma. 2.1. *Let $F_1,\dots,F_k$ denote the $k=2^{s+t}$ elements of the matrix representation of Clifford algebra $\mathcal{C}l(s,t)$ on $\mathbb{R}^k$. Then there exists a real vector $X\in\mathbb{R}^k$ such that the dimension of a linear span $\langle F_iX|i=1,\dots,k\rangle$ equals to k.*

Proof. Let us suppose, without loss of generality, that $F_1,\dots,F_k$ are the elements constructed by means of Boot periodicity as above. Then, by induction we prove

that the matrix $F=\sum_{i=1}^{k} a_i F_i,\ a_i\in\mathbb{R}$, is a square matrix that has exactly one entry a_i in each column and each row. For $\mathcal{C}l(1,0)$ and $\mathcal{C}l(0,1)$ we have

$$F=\begin{pmatrix} a_0 & a_1\\ a_1 & a_0\end{pmatrix}, F=\begin{pmatrix} a_0 & a_1\\ -a_1 & a_0\end{pmatrix},$$

respectively. For the rest of the generating cases $\mathcal{C}l(2,0)$, $\mathcal{C}l(0,2)$ and $\mathcal{C}l(1,1)$ the matrix F can be obtained in very similar way, e.g. for $\mathcal{C}l(2,0)$ we have

$$F=\begin{pmatrix} a_0 & -a_1 & a_2 & -a_3\\ -a_1 & a_0 & -a_3 & a_2\\ a_2 & a_3 & a_0 & a_1\\ a_3 & a_2 & a_1 & a_0\end{pmatrix}.$$

We now restrict to Clifford algebras of type $\mathcal{C}l(s,0),\ s>2$, and show the induction step by means of the periodicity condition

$$\mathcal{C}l(s,0)\cong\mathcal{C}l(0,s-2)\otimes\mathcal{C}l(2,0).$$

The rest of the cases according to the Clifford algebra identification above can be proved similarly and we leave it to the reader. Let $G_i,\ i=1,\ldots,l$ denote the l elements of the matrix representation of Clifford algebra $\mathcal{C}l(0,s-2)$ with the required property, i.e. the matrix $G=\sum_{i=1}^{l} g_i G_i$ is a square matrix with exactly one entry g_i in each column and each row, i.e.

$$G:=\begin{pmatrix} g_{\sigma_1(1)} & \cdots & g_{\sigma_1(l)}\\ \vdots & & \vdots\\ g_{\sigma_l(1)} & \cdots & g_{\sigma_l(l)}\end{pmatrix},$$

where σ_i are all perumtations of $\{1,\ldots,l\}$. The matrix of $\mathcal{C}l(2,0)$ is

$$H:=\begin{pmatrix} a_1 & -a_2 & a_3 & -a_4\\ -a_2 & a_1 & -a_4 & a_3\\ a_3 & a_4 & a_1 & a_2\\ a_4 & a_3 & a_2 & a_1\end{pmatrix}.$$

Then the matrix for the representation of Clifford algebra $\mathcal{C}l(s,0)$ is composed as follows:

$$F:=\begin{pmatrix} g_{\sigma_1(1)}H & \cdots & g_{\sigma_1(l)}H\\ \vdots & & \vdots\\ g_{\sigma_l(1)}H & \cdots & g_{\sigma_l(l)}H\end{pmatrix}.$$

Finally, if the matrix G has exactly one g_i in each column and each row, the matrix F is a square matrix with exactly one $a_j g_i$, where $j=1,\ldots,4, i=1,\ldots,l$ in each column and each row.

Now, let $F=\sum_{i=1}^{k} b_i F_i$ be a $k\times k$ matrix constructed as above and let e_i denote the standard basis of $\mathbb{R}^k$. Then the vector

$$v_i:=Fe_i^T$$

is the i-th column of the matrix F and thus it is composed of k different entries b_i. If the dimension of $\langle F_i X | i=1\ldots,k\rangle$ was less then k, then the vector v has to be zero and thus all b_i have to be zero. $\square$

Definition. 2.2. Let P^1M be a bundle of linear frames over M (the fiber bundle P^1M is a principal bundle over M with the structure group $GL(n,\mathbb{R})$). Reduction of the bundle P^1M to the subgroup $G \subset GL(n,\mathbb{R})$ is called a *G-structure*.

Definition. 2.3. If M is an km–dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$ then an almost Clifford manifold is given by a reduction of the structure group $GL(km,\mathbb{R})$ of the principal frame bundle of M to

$$GL(m,\mathcal{O}) = \{A \in GL(km,\mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\},$$

where $\mathcal{O}$ is an arbitrary Cliffford algebra.

In other words, an almost Clifford manifold is a smooth manifold equipped with the set of anti commuting affinors $I_i, i = 1,\dots,t,\ I_i^2 = -E$ and $J_j,\ j = 1,\dots,s,\ J_j^2 = E$ such that the free associative unitary algebra generated by $\langle I_i, J_j, E\rangle$ is isomorphically equivalent to $\mathcal{O}$. In particular, on the elements of this reduced bundle one can define affinors in the form of $F_1,\dots,F_k$ globally.

3. A-planar curves and morphisms

The concept of planar curves is a generalization of a geodesic on a smooth manifold equipped with certain structure. In [7] authors proved a set of facts about structures based on two different affinors. Following [4, 3], a manifold equipped with an affine connection and a set of affinors $A = \{F_1,\dots,F_l\}$ is called an A-structure and a curve satisfying $\nabla_{\dot c}\dot c \in \langle F_1(\dot c),\dots,F_l(\dot c)\rangle$ is called an A-planar curve.

Definition. 3.1. Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to
$A_xM \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional *A–structure.*

It is easy to see that an almost Clifford structure is not an A–structure, because the affinors in the form of $F_0,\dots,F_\ell \in A$ have to be defined only locally.

Definition. 3.2. The A–structure where A is a Clifford algebra $\mathcal{O}$ is called an almost Cliffordian manifold.

Classical concept of F–planar curves defines the *F-planar* curve as the curve $c : \mathbb{R} \to M$ satisfying the condition

$$\nabla_{\dot c}\dot c \in \langle \dot c, F(\dot c)\rangle,$$

where F is an arbitrary affinor. Clearly, geodesics are F-planar curves for all affinors, because $\nabla_{\dot c}\dot c \in \langle \dot c\rangle \subset \langle \dot c, F(\dot c)\rangle$.

Now, for any tangent vector $X \in T_xM$ we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F_i(X) | F_i \in A_xM\} \subset T_xM$$

and call it the *A–hull of the vector X*. Similarly, *A–hull of a vector field* is a subbundle in TM obtained pointwise. For example, A–hull of an almost quaternionic structure is

$$A_x(X) = \{aX + bI(X) + cJ(X) + dK(X) | a,b,c,d \in \mathbb{R}\}.$$

Definition. 3.3. Let M be a smooth manifold equipped with an A–structure and a linear connection ∇. A smooth curve $c : \mathbb{R} \to M$ is said to be *A–planar* if

$$\nabla_{\dot{c}}\dot{c} \in A(\dot{c}).$$

One can easily check that the class of connections

$$[\nabla]_A = \nabla + \sum_{i=1}^{\dim A} \Upsilon_i \otimes F_i, \tag{1}$$

where Υ_i are one forms on M, share the same class of A–planar curves, but we have to describe them more carefully for Cliffordian manifolds.

Theorem. 3.4. *Let M be a smooth manifold equipped with an almost Cliffordian structure, i.e. an A–structure, where $A = \mathcal{C}l(s,t)$, $\dim(M) \geq 2(s+t)$, and let ∇ be a linear connection such that $\nabla A = 0$. The class of connections $[\nabla]$ preserving A, sharing the same torsion and A–planar curves is isomorphic to T^*M and the isomorphism has the following form:*

$$\Upsilon \mapsto \nabla + \sum_{i=1}^{k} \epsilon_i(\Upsilon \circ F_i) \odot F_i, \tag{2}$$

where $\langle F_1, \dots, F_k\rangle = A$, $k = 2^{s+t}$, as a vector space, $\epsilon_i \in \{\pm 1\}$ and Υ is a one form on M.

Proof. First, let us consider the difference tensor

$$P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$$

and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve F_i, $i = 1, \dots, k$, the difference tensor P is Clifford linear in the second variable. By symmetry it is thus Clifford bilinear and we can procesed by induction. Let $X = \dot{c}$ and the deformation $P(X,X)$ equals $\sum_{i=1}^{k} \Upsilon_i(X)F_i(X)$ because c is A-planar with respect to ∇ and $\bar{\nabla}$ In this case we shall verify

First, for $s = 1, t = 0$

$$\begin{aligned} P(X,X) &= a(X)X + b(JX)JX, \\ P(X,X) &= J^2P(X,X) = P(JX,JX) = a(JX)JX + b(X)X. \end{aligned}$$

The difference of the first row and the second rows implies $a(X) = b(X)$ and $a(JX) = b(JX)$ because we can suppose that X, JX are linearly independent. For $s = 0, t = 1$

$$\begin{aligned} P(X,X) &= a(X)X + b(IX)IX, \\ -P(X,X) &= I^2P(X,X) = P(IX,IX) = a(IX)IX - b(X)X. \end{aligned}$$

The sum of the first row and the second row implies $a(X) = b(X)$ and $a(IX) = -b(IX)$ because we can suppose that X, IX are linearly independent.

Let us suppose that the property holds for a Clifford algebra $\mathcal{C}l(s,t)$, $k = 2^{s+t}$ i.e.

$$P(X,X) = \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(X)))F_i(X),$$

where $\epsilon_i \in \{\pm 1\}$.

For $\mathcal{C}l(s,t+1)$ we have

$$P(X,X)=\sum_{i=1}^{k}\epsilon_i(\Upsilon(F_i(X)))F_i(X)+\sum_{i=1}^{k}(\xi_i(F_iS(X)))F_iS(X),$$

and

$$S^2P(X,X)=\sum_{i=1}^{k}\epsilon_i(\Upsilon(F_i(SX)))F_i(SX)+\sum_{i=1}^{k}(\xi_i(F_i(X)))F_i(X),$$

The sum of the first row and the second rows implies

$$\epsilon_i\Upsilon(F_i(X)))=-\xi_i(F_iX) \text{ and } \epsilon_i\Upsilon(F_i(SX)))=-\xi_i(F_iSX),$$

because we can suppose that F_iX are linearly independent. The case of $\mathcal{C}l(s+1,t)$ is calculated in the same way.

Now, $P(X,X)=\sum_{i=1}^{k}\epsilon_i(\Upsilon(F_i(X)))F_i(X)$ an one shall compute

$$P(X,Y)=\frac{1}{2}(\sum_{i=1}^{k}\epsilon_i\Upsilon(F_i(X+Y))F_i(X+Y)-\sum_{i=1}^{k}\epsilon_i\Upsilon(F_i(X))F_i(X)$$
$$-\sum_{i=1}^{k}\epsilon_i\Upsilon(F_i(Y))F_i(Y)).$$

by polarization.

Assuming that vectors $F_i(X),F_i(Y)$, $i=1,\dots,k$ are linearly independent we compare the coefficients of X in the expansions of $P(sX,tY)=stP(X,Y)$ as above to get

$$s\Upsilon(sX+tY)-s\Upsilon(sX)=st(\Upsilon(X+Y)-\Upsilon(X)).$$

Dividing by s and, putting $t=1$ and taking the limit $s\to 0$, we conclude that $\Upsilon(X+Y)=\Upsilon(X)+\Upsilon(Y)$.

We have proved that the form Υ is linear in X and

$$(X,Y)\to\sum_{i=1}^{k}\epsilon_i(\Upsilon(F_i(X)))F_i(Y)+\sum_{i=1}^{k}\epsilon_i(\Upsilon(F_i(Y)))F_i(X)$$

is a symmetric complex bilinear map which agrees with $P(X,Y)$ if both arguments coincide, it always agrees with P by polarization and $\bar\nabla$ lies in the projective equivalence class $[\nabla]$.

□

4. $\mathcal{D}$–connections

Let $\mathbb{V}=\mathbb{R}^n$, $G\subset GL(\mathbb{V})=GL(n,\mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and M be a smooth manifold of dimension n.

Definition. 4.1. The *first prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is a space of symmetric bilinear mappings $t:\mathbb{V}\times\mathbb{V}\to\mathbb{V}$ such that, for each fixed $v_1\in\mathbb{V}$, the mapping $v\in\mathbb{V}\mapsto t(v,v_1)\in\mathbb{V}$ is in $\mathfrak{g}$.

Example. 4.2. A *complex structure* $(M,I),I^2=-E$, is a G-structure where $G=GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}=\{A\in\mathfrak{gl}(2n,\mathbb{R})|AI=IA\}$. The first prolongation $\mathfrak{g}^{(1)}$ is a space of symmetric bilinear mappings

$$\mathfrak{g}^{(1)}=\{t|t:\mathbb{V}\times\mathbb{V}\to\mathbb{V},t(IX,Y)=It(X,Y),t(Y,X)=t(X,Y)\}.$$

On the other hand, a *product structure* $(M,P), P^2=E$ is a G-structure where $G=GL(n,\mathbb{R})\oplus GL(n,\mathbb{R})$ with Lie algebra $\mathfrak{g}=\mathfrak{g}(n,\mathbb{R})\oplus\mathfrak{g}(n,\mathbb{R})$. The first prolongation $\mathfrak{g}^{(1)}$ is a space of symmetric bilinear mappings

$$\mathfrak{g}^{(1)}=\{t|t:\mathbb{V}_1\oplus\mathbb{V}_2\times\mathbb{V}_1\oplus\mathbb{V}_2\to\mathbb{V}_1\oplus\mathbb{V}_2, t(\mathbb{V}_i,\mathbb{V}_i)\in\mathbb{V}_i, t(\mathbb{V}_2,\mathbb{V}_1)=0\}.$$

Lemma. 4.3. *Let M be a (km)–dimensional Clifford manifold based on Clifford algebra $\mathcal{O}=\mathcal{C}l(s,t)$, $k=2^{s+t}$, $s+t>1$, $m\in\mathbb{N}$, i.e. manifold equipped with G–structure, where*

$$G=GL(m,\mathcal{O})=\{B\in GL(km,\mathbb{R})|BI_i=I_iB,\ BJ_j=J_jB\},$$

and I_i and J_j are algebra generators of $\mathcal{O}$. Then the first prolongation $\mathfrak{g}^{(1)}$ of Lie algebra $\mathfrak{g}$ of Lie group G vanishes.

Proof. Lie algebra $\mathfrak{g}$ of a Lie group G is of the form

$$\mathfrak{g}=\mathfrak{gl}(m,\mathcal{O})=\{B\in\mathfrak{gl}(km,\mathbb{R})|BI_i=I_iB,\ BJ_j=J_jB\},$$

where I_i and J_j are generators of $\mathcal{O}$, i.e. $K_{\bar{i}}K_{\bar{j}}=-K_{\bar{j}}K_{\bar{i}}$ for $K_{\bar{i}},K_{\bar{j}}\in\{I_i,J_j\}$, $\bar{i}\neq\bar{j}$. For $t\in\mathfrak{g}^{(1)}$ and $K_{\bar{i}}\neq K_{\bar{j}}$ we have equations

$$t(K_{\bar{i}}X,K_{\bar{j}}X)=K_{\bar{i}}K_{\bar{j}}t(X,X),$$
$$t(K_{\bar{i}}X,K_{\bar{j}}X)=t(K_{\bar{j}}X,K_{\bar{i}}X)=K_{\bar{j}}K_{\bar{i}}t(X,X)=-K_{\bar{i}}K_{\bar{j}}t(X,X),$$

which lead to $t(X,X)=0$. Finally from polarization

$$t(X,Y)=\frac{1}{2}(t(X+Y,X+Y)-t(X,X)-t(Y,Y))=0.\qquad\square$$

Let us shorty note, that the Example 4.2 covers Clifford manifold for $\mathcal{O}=\mathcal{C}l(0,1)$ and $\mathcal{O}=\mathcal{C}l(1,0)$. Next, suppose that there is a G-invariant complement $\mathcal{D}$ to $\partial(\mathfrak{g}\otimes\mathbb{V}^*)$ in $\mathbb{V}\otimes\wedge^2\mathbb{V}^*$:

$$\mathbb{V}\otimes\wedge^2\mathbb{V}^*=\partial(\mathfrak{g}\otimes\mathbb{V}^*)\oplus\mathcal{D},$$

where

$$\partial:\mathrm{Hom}(\mathbb{V},\mathfrak{g})=\mathfrak{g}\otimes\mathbb{V}^*\to\mathbb{V}\otimes\wedge^2\mathbb{V}^*$$

is the Spencer operator of alternation.

Definition. 4.4. Let $\pi:P\to M$ be a G-structure. A connection ω on P is called a $\mathcal{D}-$*connection* if its torsion function

$$t^\omega:P\to\mathbb{V}\otimes\wedge^2\mathbb{V}^*=\partial(\mathfrak{g}\otimes\mathbb{V}^*)\oplus\mathcal{D}$$

has values in $\mathcal{D}$.

Theorem. 4.5. [1]

1. *Any G-structure $\pi:P\to M$ admits a $\mathcal{D}$-connection ∇.*
2. *Let $\omega,\bar{\omega}$, be two $\mathcal{D}$-connections. Then the corresponding operators of covariant derivative $\nabla,\bar{\nabla}$ are related by*

$$\bar{\nabla}=\nabla+S,$$

where S is a tensor field such that for any $x\in M$, S_x belongs to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$.

Definition. 4.6. We say that a connected linear Lie group G with Lie algebra $\mathfrak{g}$ is of type k if its k-th prolongation vanishes, i.e. $\mathfrak{g}^{(k)}=0$ and $\mathfrak{g}^{(k-1)}\neq 0$. In this sense, any G-structure with Lie group G of type k is called a *G-structure of type k.*

Theorem. 4.7. [1] *Let $\pi : P \to M$ be a G-structure of type 1 and suppose that there is given a G-equivariant decomposition*

$$\mathbb{V} \otimes \wedge^2 \mathbb{V} = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus \mathcal{D}.$$

Then there exists a unique connection, whose torsion tensor(calculated with respect to a coframe $p \in P$) has values in $\mathcal{D} \subset \mathbb{V} \otimes \wedge^2 \mathbb{V}^*$.

Corollary. 4.8. *Let M be a smooth manifold equipped with a G–structure, where $G = GL(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C}l(s,t)$, $s + t > 1$, i.e. an almost Clifford manifold. Then the G–structure is of type 1 and there exists a unique $\mathcal{D}$–connection.*

5. An almost Clifordian manifold

One can see that an almost Cliffordian manifold M is given as a G–structure provided that there is a reduction of the structure group of the principal frame bundle of M to

$$G := GL(m, \mathcal{O})GL(1, \mathcal{O}) = GL(m, \mathcal{O}) \times_{Z(GL(1,\mathcal{O}))} GL(1, \mathcal{O}),$$

where $Z(G)$ is a center of G. The action of G on T_xM looks like

$$QXq, \text{ where } Q \in GL(m, \mathcal{O}), q \in GL(1, \mathcal{O}),$$

where the right action of $GL(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1, \dots, F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $GL(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $GL(m, \mathcal{O})GL(1, \mathcal{O})$ is of the form

$$\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \oplus \mathfrak{gl}(1, \mathcal{O}).$$

Let us note that the case of $\mathcal{C}l(0, 3)$ was studied in a detailed way in [2].

Remark. 5.1. Let $\mathcal{O}$ be the Clifford algebra $\mathcal{C}l(0, 2)$. For any one–form ξ on $\mathbb{V}$ and any $X, Y \in \mathbb{V}$, the elements of the form

$$\begin{aligned} S^\xi(X, Y) &= -\xi(X)Y - \xi(Y)X + \xi(I_1X)I_1Y + \xi(I_1Y)I_1X + \xi(I_2X)I_2Y \\ &\quad + \xi(I_2Y)I_2X + \xi(I_1I_2X)I_1I_2Y + \xi(I_1I_2Y)I_1I_2X \end{aligned}$$

belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m, \mathcal{O})GL(1, \mathcal{O})$.

Proof. We fix $X \in \mathbb{V}$ and define $S^\xi_X := S^\xi(X, Y) : \mathbb{V} \to \mathbb{V}$. We have to prove that $S^\xi_X(I_iY) = I_iS^\xi_X(Y) + \sum_{l=1}^4 a_lF_l(Y)$, for $i = 1, 2$ and $S^\xi_X(Y) = S^\xi_Y(X)$. We compute directly for any X and for I_1

$$\begin{aligned} S^\xi_X(I_1Y) &= -\xi(X)I_1Y - \xi(I_1Y)X - \xi(I_1X)Y - \xi(Y)I_1X - \xi(I_2X)I_1I_2Y \\ &\quad - \xi(I_1I_2Y)I_2X + \xi(I_1I_2X)I_2Y + \xi(I_2Y)I_1I_2X \\ &= -\xi(I_1Y)X - \xi(Y)I_1X - \xi(I_1I_2Y)I_2X \\ &\quad + \xi(I_2Y)I_1I_2X + \sum_{l=1}^4 a_lF_l(Y). \end{aligned}$$

On the other hand,

$$\begin{aligned}I_1S^{\xi}_X(Y) &= -\xi(X)I_1Y-\xi(Y)I_1X-\xi(I_1X)Y-\xi(I_1Y)X+\xi(I_2X)I_1I_2Y\\&\quad+\xi(I_2Y)I_1I_2X-\xi(I_1I_2X)I_2Y-\xi(I_1I_2Y)I_2X\\&= -\xi(Y)I_1X-\xi(I_1Y)X+\xi(I_2Y)I_1I_2X\\&\quad-\xi(I_1I_2Y)I_2X+\sum_{l=1}^{4}a_lF_l(Y)\end{aligned}$$

and

$$\begin{aligned}&S^{\xi}_X(I_1Y)-I_1S^{\xi}_X(Y)=\\&\qquad=-\xi(I_1Y)X-\xi(Y)I_1X-\xi(I_1I_2Y)I_2X+\xi(I_2Y)I_1I_2X\\&\qquad-(-\xi(Y)I_1X-\xi(I_1Y)X+\xi(I_2Y)I_1I_2X-\xi(I_1I_2Y)I_2X)\\&\qquad+\sum_{l=1}^{4}\bar{a}_lF_l(Y)=\sum_{l=1}^{4}\bar{a}_lF_l(Y).\end{aligned}$$

By the same process for I_2 we obtain

$$\begin{aligned}&S^{\xi}_X(I_2Y)-I_2S^{\xi}_X(Y)=\\&\qquad=-\xi(I_2Y)X+\xi(I_1I_2Y)I_1X-\xi(Y)I_2X-\xi(I_1Y)I_1I_2X\\&\qquad-(-\xi(Y)I_2X-\xi(I_1Y)I_1I_2X-\xi(I_2Y)X+\xi(I_1I_2Y)I_1X)\\&\qquad+\sum_{l=1}^{4}\bar{a}_lF_l(Y)=\sum_{l=1}^{4}\bar{a}_lF_l(Y).\end{aligned}$$

Finally, we have to prove the symmetry, but this is obvious. □

Lemma. 5.2. *Let $\mathcal{C}l(s,t)$ be the Clifford algebra, $n=s+t$, and let us denote by F_i the affinors obtained from the generators of $\mathcal{C}l(s,t)$. Then there exist $\varepsilon_i\in\{\pm1\}$, $i=1,...,n$ such that for $A\in V^*$, the tensor $S^A\in V\times V\to V$ defined by*

$$S^A(X,Y)=\sum_{i=1}^{n}\varepsilon_iA(F_iX)F_iY,\ \ X,Y\in V, \tag{3}$$

satisfies the identity

$$S^A(I_jX,Y)-I_jS^A(X,Y)=0 \tag{4}$$

for all algebra generators I_j of $\mathcal{C}l(s,t)$.

Proof. Let us consider the gradation of the Clifford algebra $\mathcal{C}l(s,t)=\mathcal{C}l^0\oplus\mathcal{C}l^1\oplus ...\oplus\mathcal{C}l^n$ with respect to the generators of $\mathcal{C}l(s,t)$. Then we can define gradually: for $E\in\mathcal{C}l^0$ we choose $\varepsilon=1$. If the identity (4) should be satisfied for the terms in (3), then it must hold

$$\varepsilon_0A(I_jX)Y=\varepsilon_iA(I_jX)I_jI_jY \text{ for all } I_j,$$

i.e.

$$\varepsilon_i=\begin{cases}1 & \text{for } I_j^2=1,\\-1 & \text{for } I_j^2=-1.\end{cases}$$

For $F_i\in\mathcal{C}l^v$ the following equality holds:

$$\varepsilon_iA(F_iI_jX)F_iY=\varepsilon_kA(F_kX)I_jF_kY$$

and thus $F_i = I_j F_k$. Note that F_k can be an element of both $\mathcal{C}l^{v+1}$ and $\mathcal{C}l^{v-1}$. W.l.o.g. we choose I_j such that $F_k \in \mathcal{C}l^{v+1}$. Now two possibilities can appear: Either

$$F_i I_j = F_k, \tag{5}$$

which leads to $I_j F_k I_j = F_k$ and thus $\varepsilon_k = \varepsilon_i$, or

$$F_i I_j = -F_k, \tag{6}$$

which leads to $I_j F_k I_j = -F_k$ and thus $\varepsilon_k = -\varepsilon_i$.

This concludes the definition of ε_i such that the identity (4) holds. To prove the consistency, we have to show that the value of ε_k does not depend on I_j, i.e. for the generators I such that $I^2 = 1$ and J such that $J^2 = -1$, the resulting coefficient ε_k obtained after two consequent steps of the algorithm with the alternate use of both I and J, does not depend on the order. Thus let us consider the following cases:

(a) $F_i = IF_k$, which results into the possibilities $IF_k I = F_k$, see (5), which leads to $\varepsilon_k = \varepsilon_i$, or $IF_k I = -F_k$, see (6), which leads to $\varepsilon_k = -\varepsilon_i$.
(b) $F_j = JF_k$,which similarly leads to either $JF_k J = F_k$ implying $\varepsilon_k = \varepsilon_j$, or $JF_k J = -F_k$ implying $\varepsilon_k = -\varepsilon_j$.

Applying the processes (a) and (b) alternately we obtain:

$$JIF_k J = \begin{cases} -IF_k & \Rightarrow \varepsilon_l = -\varepsilon_i \\ IF_k & \Rightarrow \varepsilon_l = \varepsilon_i \end{cases}$$

for $F_l = JIF_k$ and

$$IJF_k I = \begin{cases} -JF_k & \Rightarrow \varepsilon_l = -\varepsilon_j \\ JF_k & \Rightarrow \varepsilon_l = \varepsilon_j \end{cases}$$

for $F_l = IJF_k$. Obviously, the corresponding cases give the same result of ε_k. □

Theorem. 5.3. *Let $\mathcal{O}$ be the Clifford algebra $\mathcal{C}l(s,t)$. For any one–form ξ on $\mathbb{V}$ and any $X, Y \in \mathbb{V}$, the elements of the form*

$$S_X^\xi(Y) = \sum_{i=1}^{k} \epsilon_i(\xi(F_i X)F_i Y + \xi(F_i Y)F_i X), \;\; k = 2^{s+t},$$

where the coefficients ϵ_i depend on the type of $\mathcal{O}$, belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$.

Proof. One can easily see that S^ξ is symmetric and we have to prove the second condition, i.e. $S_X^\xi I_i Y - I_i S_X^\xi Y \in \mathcal{O}(Y)$, i.e.

$$\begin{aligned} S_X^\xi I_i Y - I_i S_X^\xi Y = & \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_j X) F_j Y + \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_j Y) F_j X \\ & - \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_j X) F_j Y - \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_j Y) F_j TX. \end{aligned}$$

From Lemma 5.2 we have

$$S_X^\xi I_i Y - I_i S_X^\xi Y = \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_j X) F_j Y - \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_j Y) F_j TX = \sum_{j=0}^{k} \psi_i F_j Y.$$

□

Corollary. 5.4. *Let M be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}l(s,t)$, where $\dim(M) \geq 2(s+t)$, i.e. smooth manifold equipped with G–structure, where $G = GL(n,\mathcal{O})GL(1,\mathcal{O})$ or equivalently A–structure where $A = \mathcal{O}$. Then the class of $\mathcal{D}$–connections preserving A and sharing the same A–planar curves is isomorphic to $(\mathbb{R}^{km})^*$.*

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